

Langmuir dark solitons in dense ultrarelativistic electron-positron gravito-plasma in pulsar magnetosphere

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Abstract

Nonlinear propagation of electrostatic modes in ultrarelativistic dense electron-positron gravito-plasma at the polar cap region of pulsar magnetosphere is considered. A nonlinear Schrödinger equation is obtained from the reductive perturbation method which predicts the existence of Langmuir dark solitons. Relevance of the propagating dark solitons to the pulsar radio emission is discussed.

Keywords: Electron-positron plasma; Langmuir solitons; pulsar magnetosphere.

I. Introduction

Pulsars are celestial sources that believe to be rotating neutron stars producing light-house like beams of radio emissions from the magnetic poles. As shown by Goldreich and Julian (1969) the rotating magnetic dipole produces a quadrupole electric field whose component parallel to the open magnetic field lines at the poles extracts particles very effectively from neutron star surface and accelerates them to highly relativistic energies. Thus, the magnetosphere is filled with plasma which shields the electric field. Complete shielding is established when the net charge reaches n_{GJ} - the Goldreich-Julian charge density. The Lorentz factors of the accelerated particles reach about 10^6 and they emit hard curvature radiations that propagate at a sufficient angle to the magnetic field, so that significant pair production of electron-positron can occur (Erber 1966). It is commonly accepted that the newly created particles produce more pairs by emitting energetic synchrotron or curvature radiation. As a result an avalanche of secondary particles populates the magnetosphere with densities $10^4 n_{GJ}$ (Ruderman and Sutherland, 1975). Here, we extend our earlier research on pulsar microstructure, soliton formation, wakefield accelerations, gravitational waves, and growing modes (Mofiz, et al. 1985-2011) to account the pair ultrarelativistic pressure and the gravity

The paper is organized as follows. Section II describes the fluid model of the dense ultrarelativistic electron-positron plasma under gravity. Considering a Lorentz invariant frame moving with group velocity of the wave, a Nonlinear Schrödinger Equation (NLSE) is derived using the reductive perturbation method (Gardner and

Morikawa, 1960). A linear dispersion relation is obtained showing the existence of Langmuir waves under gravity and with ultrarelativistic temperature for wave propagation. The solution of NLSE shows the generation of Langmuir dark solitons. Results are discussed in Sec. III. Finally, Sec. IV concludes the paper.

II. The Mathematical Model

We consider two-fluid magnetohydrodynamic (MHD) equations to describe the electron-positron plasma in the pulsar magnetosphere of the neutron star. The equations are the usual continuity and momentum balance equations for the plasma species, electrons and positrons, supplemented by the Poisson's equation for electrostatic wave propagation. Thus, the required set of equations are as follows (Mofiz and Ahmedov, 2000):

$$\frac{\partial}{\partial t} (\gamma_s n_s) + \nabla \cdot \left(\sqrt{1 - \frac{r_g}{r}} \gamma_s n_s \mathbf{v}_s \right) = 0, \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial t} (\gamma_s \mathbf{v}_s) + \sqrt{1 - \frac{r_g}{r}} \mathbf{v}_s \cdot \nabla (\gamma_s \mathbf{v}_s) \\ = -\frac{q_s}{m} \nabla \phi - \frac{1}{m \gamma_s n_s} \nabla p_s, \end{aligned} \quad (2)$$

$$\nabla \cdot \left(\frac{1}{\sqrt{1 - \frac{r_g}{r}}} \nabla \phi \right) = -4\pi \sum_s \gamma_s q_s n_s, \quad (3)$$

where, $\gamma_s = 1/\sqrt{1 - v_s^2/c^2}$, n_s , \mathbf{v}_s , and q_s are respectively the particle number density, particle velocity, and particle charge of plasma species s ; $q_s = -e$ for $s = e$ (electron) and $q_s = +e$ for $s = e^+$ (positron); ϕ is

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the electrostatic potential, r_g is the Schwarzschild radius of the neutron star; m is the electron/positron mass; e is the absolute value of the electronic charge. In Eq. (2), the pressure p_s is given by the expression for the ultrarelativistic pressure (Chandrasekhar, 1938): $p_s = n_0 k_B T (n_s/n_0)^{4/3}$, where $T_e = T_p = T$ has been assumed; k_B is the Boltzmann constant, n_0 is the equilibrium particle number density.

In the polar cap region of the pulsar, we consider $\theta = 0$, $\nabla = \hat{z}\partial/\partial z$, $\mathbf{v}_s = v_{sz}\hat{z}$ and adopt the following normalization of different quantities: $z \rightarrow z\omega_{pe}/c$, $t \rightarrow t\omega_{pe}$, $n_s \rightarrow n_s/n_0$, $u_s \rightarrow v_{sz}/c$, $\phi \rightarrow e\phi/mc^2$, $\sigma_T \rightarrow k_B T/mc^2$, $r_g \rightarrow r_g\omega_{pe}/c$, and $v_g \rightarrow v_g/c$, where $\omega_{pe} = (4\pi n_0 e^2/m)^{1/2}$ is the electron plasma frequency, c is the speed of light. To study the nonlinear dynamics, we consider the following stretched coordinates in the moving frame (Melikidze et al. 2000):

$$\xi = \epsilon\gamma_0 (z - v_g t), \quad (4)$$

and

$$\tau = \epsilon^2\gamma_0 (t - v_g z), \quad (5)$$

where, v_g is the group velocity, $\gamma_0 = \gamma_e = \gamma_p$ is the average relativistic Lorentz factor and is given by the following expression: $\gamma_0 = (1 - v_g^2)^{-1/2}$, ϵ is a small quantity, the perturbation parameter with $\epsilon < 1$. With the transformations, given by Eqs. (4) and (5), the derivatives $\partial/\partial t$ and $\partial/\partial z$ transform to $\partial/\partial t \rightarrow \partial/\partial t - \epsilon\gamma_0 v_g \partial/\partial \xi + \epsilon^2\gamma_0 \partial/\partial \tau$ and $\partial/\partial z \rightarrow \partial/\partial z + \epsilon\gamma_0 \partial/\partial \xi - \epsilon^2\gamma_0 v_g \partial/\partial \tau$ respectively. With the above considerations, the set of equations, Eqs. (1)-(3) take the following forms:

$$\begin{aligned} \frac{\partial n_s}{\partial t} + \frac{\partial}{\partial z} (g(z)n_s u_s) \\ + \epsilon \left[-\gamma_0 v_g \frac{\partial n_s}{\partial \xi} + \gamma_0 \frac{\partial}{\partial \xi} (g(z)n_s u_s) \right] \\ + \epsilon^2 \left[\gamma_0 \frac{\partial n_s}{\partial \tau} - \gamma_0 v_g \frac{\partial}{\partial \tau} (g(z)n_s u_s) \right] = 0, \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{\partial u_s}{\partial t} + g(z)u_s \frac{\partial u_s}{\partial z} + \frac{q_s}{e\gamma_0} \frac{\partial \phi}{\partial z} + \frac{4\sigma_T}{3\gamma_0^2} n_s^{-2/3} \frac{\partial n_s}{\partial z} \\ + \epsilon \left[-\gamma_0 v_g \frac{\partial u_s}{\partial \xi} + g(z)u_s \gamma_0 \frac{\partial u_s}{\partial \xi} \right. \\ \left. + \frac{q_s}{e} \frac{\partial \phi}{\partial \xi} + \frac{4\sigma_T}{3\gamma_0} n_s^{-2/3} \frac{\partial n_s}{\partial \xi} \right] \\ + \epsilon^2 \left[\gamma_0 \frac{\partial u_s}{\partial \tau} - g(z)\gamma_0 v_g u_s \frac{\partial u_s}{\partial \tau} \right. \\ \left. - \frac{q_s v_g}{e} \frac{\partial \phi}{\partial \tau} - \frac{4\sigma_T v_g}{3\gamma_0} n_s^{-2/3} \frac{\partial n_s}{\partial \tau} \right] = 0, \end{aligned} \quad (7)$$

$$\frac{\partial}{\partial z} \left(\frac{1}{g(z)} \frac{\partial \phi}{\partial z} \right) - \gamma_0 (n_e - n_p)$$

$$\begin{aligned} + \epsilon \left[\gamma_0 \frac{\partial}{\partial \xi} \left(\frac{1}{g(z)} \frac{\partial \phi}{\partial z} \right) \right. \\ \left. + \frac{\partial}{\partial z} \left(\frac{\gamma_0}{g(z)} \frac{\partial \phi}{\partial \xi} \right) \right] \\ + \epsilon^2 \left[-\gamma_0 v_g \frac{\partial}{\partial \tau} \left(\frac{1}{g(z)} \frac{\partial \phi}{\partial z} \right) \right. \\ \left. - \gamma_0 v_g \frac{\partial}{\partial z} \left(\frac{1}{g(z)} \frac{\partial \phi}{\partial \tau} \right) \right. \\ \left. + \gamma_0 \frac{\partial}{\partial \xi} \left(\frac{\gamma_0}{g(z)} \frac{\partial \phi}{\partial \xi} \right) \right] \\ + \epsilon^3 \left[-2\gamma_0^2 \frac{\partial}{\partial \xi} \left(\frac{v_g}{g(z)} \frac{\partial \phi}{\partial \tau} \right) \right. \\ \left. - \gamma_0^2 v_g \frac{\partial}{\partial \tau} \left(\frac{1}{g(z)} \frac{\partial \phi}{\partial \xi} \right) \right] \\ + \epsilon^4 \left[\gamma_0^2 v_g^2 \frac{\partial}{\partial \tau} \left(\frac{1}{g(z)} \frac{\partial \phi}{\partial \tau} \right) \right] = 0, \end{aligned} \quad (8)$$

where, the factor $g(z) = (1 - r_g/z)^{1/2}$ accounts for the gravitational effect. Now we expand the quantities n_s , u_s , ϕ as

$$\begin{aligned} n_s = 1 + \epsilon^2 n_{s0} \\ + \sum_{l=1}^{\infty} \epsilon^l \left(n_{sl} e^{il(kz - \omega t)} + n_{sl}^* e^{-il(kz - \omega t)} \right), \end{aligned} \quad (9)$$

$$u_s = \epsilon^2 u_{s0} + \sum_{l=1}^{\infty} \epsilon^l \left(u_{sl} e^{il(kz - \omega t)} + u_{sl}^* e^{-il(kz - \omega t)} \right), \quad (10)$$

$$\phi = \epsilon^2 \phi_0 + \sum_{l=1}^{\infty} \epsilon^l \left(\phi_l e^{il(kz - \omega t)} + \phi_l^* e^{-il(kz - \omega t)} \right). \quad (11)$$

Here,

$(n_{s0}, u_{s0}, \phi_0, n_{sl}, u_{sl}, \phi_l) \equiv A^{(1)} + \epsilon A^{(2)} + \epsilon^2 A^{(3)} + \dots$ are functions of stretched coordinates (ξ, τ) .

II. A. Linear Dispersion Relation for the Langmuir Wave

Now considering $|1/g(z) \cdot dg(z)/dz| \ll k$ for the first harmonic ($l = 1$) in the first order ($\epsilon = 1$), we have the following equations for the first-order quantities:

$$-i\omega n_{s1}^{(1)} + ikg(z)u_{s1}^{(1)} = 0, \quad (12)$$

$$-i\omega u_{s1}^{(1)} + \frac{4ik\sigma_T}{3\gamma_0^2} n_{s1}^{(1)} + \frac{ikq_s}{e\gamma_0} \phi_1^{(1)} = 0, \quad (13)$$

$$-\frac{k^2}{g(z)} \phi_1^{(1)} - \gamma_0 (n_{e1}^{(1)} - n_{p1}^{(1)}) = 0. \quad (14)$$

Eliminating $u_{s1}^{(1)}$ from Eqs. (12) and (13), we obtain the following equation relating $n_{s1}^{(1)}$ and $\phi_1^{(1)}$:

$$n_{s1}^{(1)} = -\frac{k^2 g(z)}{-\omega^2 + 4\sigma_T g(z)k^2/3\gamma_0^2} \frac{q_s}{e\gamma_0} \phi_1^{(1)}, \quad (15)$$

from which we obtain

$$n_{e1}^{(1)} - n_{p1}^{(1)} = \frac{2k^2 g(z)}{-\omega^2 + 4\sigma_T g(z)k^2/3\gamma_0^2} \frac{1}{\gamma_0} \phi_1^{(1)}. \quad (16)$$

Using Eq. (16) into Eq. (14), we obtain the following linear dispersion relation:

$$\omega^2 = 2g^2(z) + \frac{4\sigma_T g(z)}{3\gamma_0^2} k^2, \quad (17)$$

which in the dimensional form is

$$\omega^2 = \omega_{pe}^2 g^2(z) + \frac{2}{3} k^2 v_{th}^2 g(z), \quad (18)$$

with $v_{th}^2 = \frac{2k_B T}{m\gamma_0^2}$. The group velocity $v_g = \partial\omega/\partial k$ is obtained from the linear dispersion relation Eq. (17) as:

$$v_g = \frac{4\sigma_T g(z)}{3\gamma_0^2} \frac{k}{\omega}. \quad (19)$$

The same expression for v_g is also obtained from the compatibility condition and shown in the Appendix A.

The group dispersion is found to be

$$v_g' = \frac{dv_g}{dk} = \frac{\frac{4}{3} \frac{\sigma_T g(z)}{\gamma_0^2} - v_g^2}{\omega}. \quad (20)$$

Eq.(17) represents the dispersion relation for Langmuir waves in ultrarelativistic e, e^+ plasma under gravity. Pair production in the polar cap region of pulsar magnetosphere occurs through curvature radiation which happens for $\mathcal{E}_{\parallel} \gg m_e c^2$, where \mathcal{E}_{\parallel} is the energy of electron along the magnetic field. It is estimated that for cascade generation of pair plasma $\gamma_0 \sim 10^6 - 10^7$, $\mathcal{E}_{\parallel} \sim 10^{12} - 10^{13} eV$ (Beskin et al. 1993). Here, $\gamma_0 = \mathcal{E}_{\parallel}/m_e c^2$, then considering $\mathcal{E}_{\parallel} = k_B T$, we find $\gamma_0 = k_B T/m_e c^2 \equiv \sigma_T$. Using Eqs.(17),(19) and Eq.(20), we perform an analysis of the dispersion relation, group velocity and group dispersion of Langmuir waves at the ultrarelativistic temperature of the e, e^+ plasma under gravity. The analysis is shown graphically in Fig.1-4, respectively.

II.B Nonlinear Evolution Equation for the Langmuir Wave

Finding the zeroth harmonic and second harmonic of the second order quantities in terms of the first harmonic of the first order quantities and using these into the first harmonic of the third order quantities, we easily obtain the following NLSE for the evolution of the potential

$$i \frac{\partial a}{\partial \tau} + P \frac{\partial^2 a}{\partial \xi^2} + Q |a|^2 a = 0, \quad (21)$$

where $a \equiv \phi_1^{(1)}$, and the coefficients P and Q are given by the following expressions:

$$P = \frac{\frac{4\sigma_T g(z)}{3\gamma_0^2} - v_g^2}{\frac{2}{\gamma_0} \left[\omega - \frac{v_g}{kg^3(z)} \{2\omega^2 + v_g k\omega + 2g^2(z)\} \right]}, \quad (22)$$

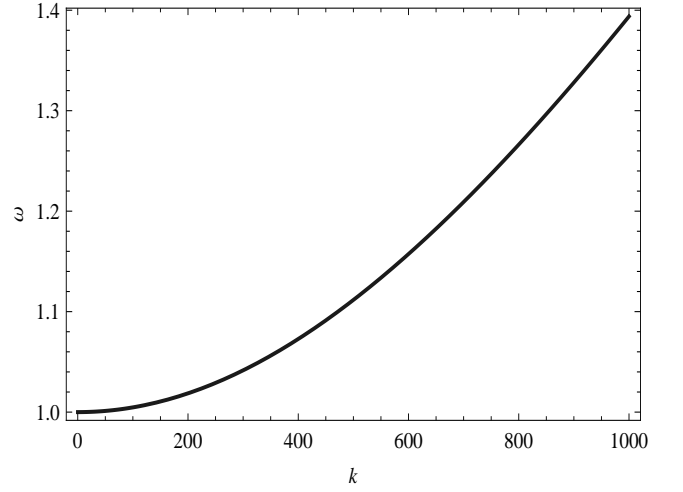


FIG. 1: The variation of the normalized frequency ω of the linear Langmuir wave with respect to the normalized pump wavenumber k for different values of plasma parameters : $r_g = 1$, $z = 2$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

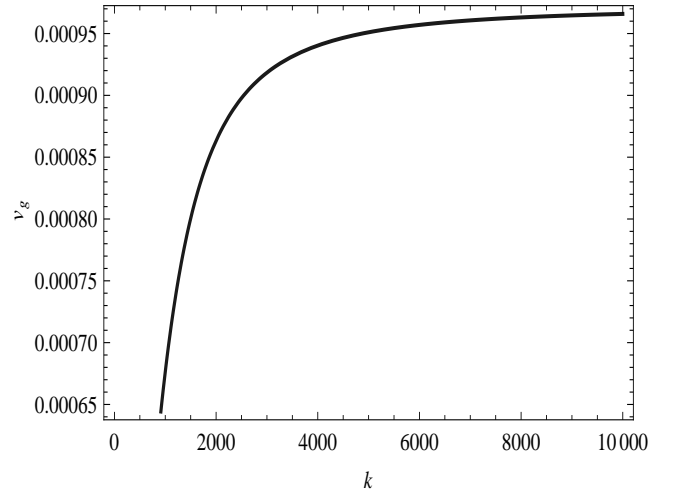


FIG. 2: The variation of the normalized group velocity v_g of the linear Langmuir wave with respect to the normalized pump wavenumber k for different values of plasma parameters : $r_g = 1$, $z = 2$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

$$Q = \frac{\frac{g^2(z)}{\gamma_0} \left[\frac{\omega}{k} f_{11} + h_{11} \right]}{\frac{2}{\gamma_0} \left[\omega - \frac{v_g}{kg^3(z)} \{2\omega^2 + v_g k\omega + 2g^2(z)\} \right]}, \quad (23)$$

where

$$f_{11} = -\frac{\omega k}{g^2(z)\gamma_0} \left[b_1 \left(1 + \frac{kv_g}{\omega} \right) + 2b_2 \right] + \frac{3\omega k^5}{4g^4(z)\gamma_0^3}, \quad (24)$$

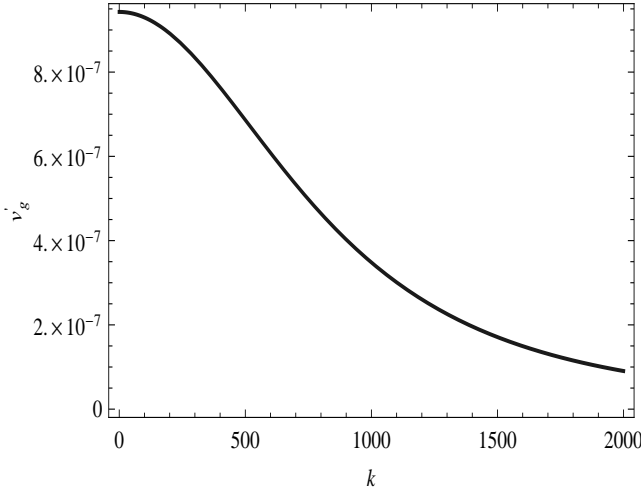


FIG. 3: The variation of the normalized group dispersion v'_g of the linear Langmuir wave with respect to the normalized pump wavenumber k for different values of plasma parameters : $r_g = 1$, $z = 2$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

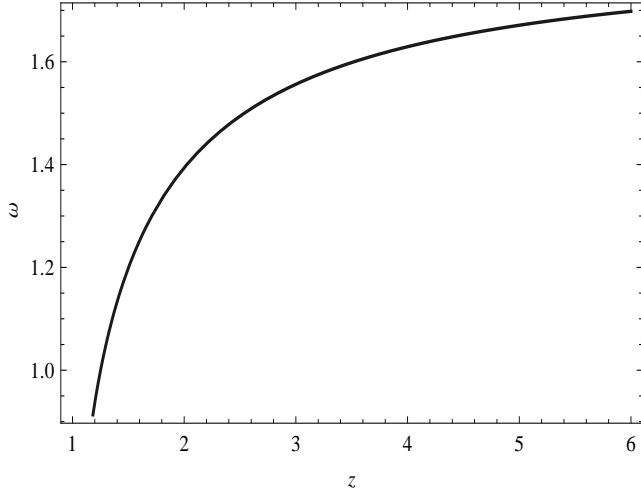


FIG. 4: The variation of the normalized frequency ω of the linear Langmuir wave with respect to the normalized distance z for parameters : $r_g = 1$, $k = 1000$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

$$h_{11} = -\frac{\omega^2}{g^2(z)\gamma_0} \left[\frac{kv_g}{\omega} \frac{b_1}{g(z)} + b_2 - \frac{3k^4}{4g^3(z)\gamma_0^2} \right] + \frac{8\sigma_T k^2}{9g^2(z)\gamma_0^3} (b_1 + b_2), \quad (25)$$

with

$$b_1 = -\frac{1}{\frac{4\sigma_T g(z)}{3\gamma_0^2} - v_g^2} \left[g^2(z) \left(\frac{\omega k}{2g^2(z)\gamma_0} \right)^2 - \frac{2\sigma_T k^4}{9g(z)\gamma_0} + \frac{v_g \omega k^3}{2g^2(z)\gamma_0^2} \right], \quad (26)$$

$$b_2 = \frac{3\omega^2 k^4}{16g^4(z)\gamma_0^2} - \frac{\sigma_T k^4}{9g^2(z)\gamma_0^3}. \quad (27)$$

Here, the coefficient P can be written as

$$P = \frac{1}{2\alpha} v'_g, \quad (28)$$

where,

$$\alpha = \frac{1}{\gamma_0} \left[1 - \frac{4\sigma_T}{\gamma_0^2 g^2(z)} \right], \quad (29)$$

represents the effects of ultrarelativistic temperature and gravity, neglecting of which we recover the usual results. The coefficients P and Q appearing in the NLSE, given by Eqs. (22), (23) are known as the dispersion and nonlinear coefficients, respectively. The signs of P and Q determine whether the slowly varying wave pulse is stable or not (Lighthill condition; Lighthill, 1967). If the signs of P and Q are such that $PQ < 0$, the wave pulse is modulationally stable and the corresponding solution of the NLSE is called the dark soliton. On the other hand, if $PQ > 0$, then the pulse may be modulationally unstable and the solution of the NLSE in this case is called the bright soliton. Graphically, we study the nature of P and Q for continuous values of the wave number k with particular values of plasma parameters, which are shown graphically in Fig.5-6, respectively.

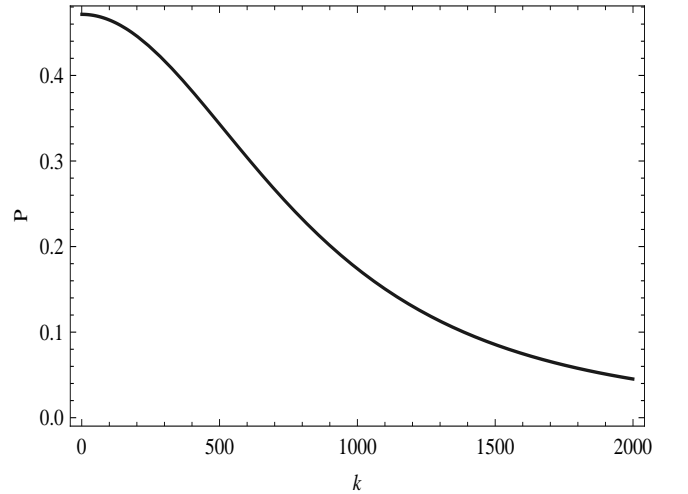


FIG. 5: The variation of the dispersion coefficient P of the linear Langmuir wave with respect to the normalized pump wavenumber k for different values of plasma parameters : $r_g = 1$, $z = 2$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

From the graphical analysis, we find that $PQ < 0$. Then applying the standard technique, and by taking $a(\xi, \tau) = a(\xi) \exp[i(K\xi - \Omega\tau)]$, the following solution of

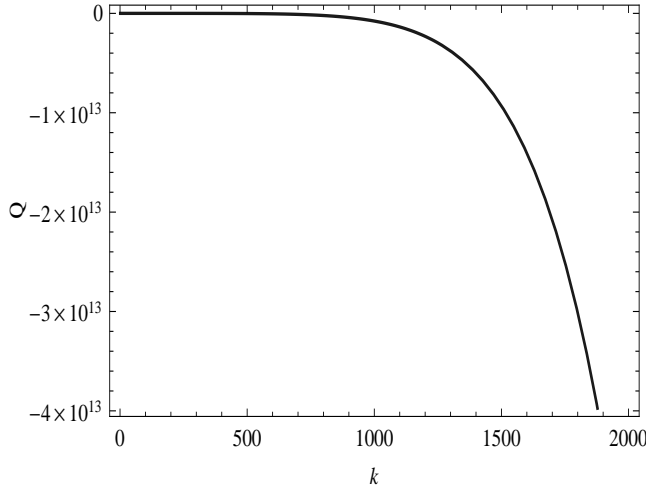


FIG. 6: The variation of the nonlinear coefficient Q of the linear Langmuir wave with respect to the normalized pump wavenumber k for different values of plasma parameters: $r_g = 1$, $z = 2$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

the NLSE, Eq. (21) (Mofiz,2007) is easily obtained:

$$a(\xi, \tau) = a_0 \tanh \left[\left| \frac{Q}{4P} \right|^{1/2} a_0 \xi \right] \exp[i(K\xi - \Omega\tau)] \quad (30)$$

Here, $K\xi - \Omega\tau$ is the modulation phase with $K (\ll k)$ and $\Omega (\ll \omega)$, respectively. Eq.(30) represents a dark soliton with amplitude

$$a_0 = \left| \frac{\Omega + PK^2}{Q} \right|^{1/2} \text{ and width } \delta = \left| \frac{4P}{Qa_0^2} \right|^{1/2}, \text{ respectively.}$$

The dark soliton (Eq.(30)) in the ultrarelativistic e, e^+ plasma is shown graphically in Fig.7.

III. Results and Discussion

In this section, we analyze the linear dispersion as well as the nonlinear Langmuir dark soliton in the pulsar magnetosphere. Eq.(17) shows that the Langmuir frequency depends on ultrarelativistic temperature and it is red-shifted due to gravity near the Schwarzschild radius. Similarly, the group velocity and group dispersion, shown by Eq.(19) and Eq. (20), are also depend on temperature and gravity.

For numerical appreciation of the dark soliton, we consider the two cases of ultrarelativistic temperatures: $6 \times 10^{10} K - 1.8 \times 10^{11} K$ (Crab pulsar) with the corresponding energies $5 - 15 MeV$ (Nanobashvili,2004) and $5 \times 10^{15} K - 5 \times 10^{16} K$ (x-ray pulsar) with the corresponding energies $10^{12} - 10^{13} eV$ (Beskin et al., 1993).

The solution of the NLSE (Eq.(21)) is a stable dark soliton (Eq.(30)) whose amplitude and width depend on temperature. The amplitude is increased and the width is decreased with the increase of ultrarelativistic

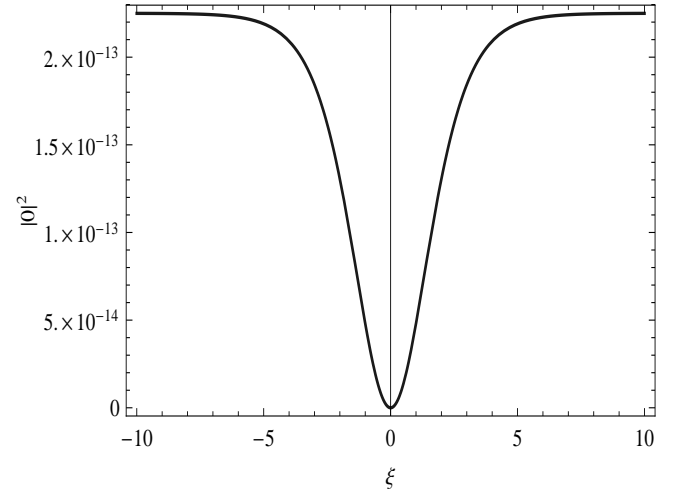


FIG. 7: Dark soliton in ultrarelativistic e, e^+ plasma at the polar cap region of pulsar magnetosphere. The parameters are: $r_g = 1$, $z = 2$, $k = 1000$, $\Omega = 0$, $K = 1$, $\gamma_0 = \sigma_T = 10^6$ with the corresponding temperature $T = 5 \times 10^{15} K$.

temperature. Thus, stable spiky Langmuir solitons are possible in the ultrarelativistic electron-positron plasma.

IV. Conclusion

To summarize, we have investigated the nonlinear propagation of electrostatic modes in a dense ultrarelativistic electron-positron gravito-plasma at the polar cap region of pulsar magnetosphere. A multiscale perturbation analysis of the fluid equations shows that stable dark Langmuir solitons are produced due to the balance of dispersion and nonlinearity in the wave propagation. As the amplitude of the soliton increases and width of the soliton decreases with the increase of ultrarelativistic temperature, so spiky stable dark Langmuir solitons may propagate along the open field lines of the pulsar magnetosphere, which may have some relation with pulsar radio emission and its microstructure.

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Appendix A: The compatibility condition

It can be shown that the 1st harmonic of the 2nd-order electron and positron densities can be found to be

$$n_{s1}^{(2)} = \frac{q_s k^2}{2eg^2(z)\gamma_0} \phi_1^{(2)} + \frac{i\gamma_0}{2g^2(z)}$$

$$\times \left[\left(\omega v_g - \frac{4\sigma_T g(z)k}{3\gamma_0^2} \right) \frac{\partial n_{s1}^{(1)}}{\partial \xi} + g(z) (kv_g - \omega) \frac{\partial u_{s1}^{(1)}}{\partial \xi} \right]$$

$$- \frac{iq_s k}{2eg(z)} \frac{\partial \phi_1^{(1)}}{\partial \xi}.$$

After finding $n_{e1}^{(2)} - n_{p1}^{(2)}$ and substituting it in the following 1st-harmonic of the 2nd-order part of the Poisson's equation:

$$- \frac{k^2}{g(z)} \phi_1^{(2)} - \gamma_0 \left(n_{e1}^{(2)} - n_{p1}^{(2)} \right) + \frac{2ik\gamma_0}{g(z)} \frac{\partial \phi_1^{(1)}}{\partial \xi} = 0,$$

we obtain the following compatibility condition:

$$v_g = \frac{4\sigma_T g(z)k}{3\gamma_0^2} \frac{1}{\omega},$$

which is exactly the same as the expression of the group velocity, Eq. (19), obtained by differentiating ω with respect to k from the linear dispersion relation, Eq. (17).

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